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Polynomial Approximation of an Entire Function and Generalized Orders

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1. INTRODUCTION

Let $f(x)$ be a real or complex-valued continuous function defined on $[-1, 1]$ and let

$$E_n(f) = \inf_{p \in \pi_n} \| f - p \|, \qquad n = 0, 1,... \tag{1.1}$$

where the norm is the sup norm on $[-1, 1]$ and π_n denotes the set of all polynomials p of degree at most n. Bernstein ([2, p. 118]; see also [5, pp. 76–78; 6, pp. 90–94]) proved that

$$\lim_{n \to \infty} E_n^{1/n}(f) = 0 \tag{1.2}$$

if and only if $f(x)$ is the restriction to $[-1, 1]$ of an entire function $f(z)$. Varga [15] obtained results giving the order and type of this entire function. Reddy [7, 8] studied the order, the lower order, and the different types (logarithmic type, lower type), and Juneja [4] studied the lower order. These authors define the order and the lower order by considering the ratio $l_j M(r)/l_1 r$ ($j \geqslant 2$; see remarks in this section; see also [9, 12; 7, Section 1]). In this paper we define the generalized order $\rho(\alpha, \beta, f)$ and the generalized lower order $\lambda(\alpha, \beta, f)$ on any entire function f and extend some known results on entire functions of infinite order. Our definition of $\rho(\alpha, \beta, f)$ is essentially due to Seremeta ([11: Theorem 1]; see also [1]).

Let L^0 denote the class of functions h satisfying the following conditions (H, i) and (H, ii);

(H, i) $h(x)$ is defined on $[a, \infty)$ and is positive strictly increasing, differentiable and tends to ∞ as $x \to \infty$.

(H, ii)
$$\lim_{x \to \infty} \frac{h((1 + 1/\psi(x))x)}{h(x)} = 1,$$

for every function $\psi(x)$ such that $\psi(x) \to \infty$ as $x \to \infty$.

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Let Λ denote the class of functions h satisfying conditions (H, i) and (H, iii);

(H, iii) $$\lim_{x\to\infty} \frac{h(cx)}{h(x)} = 1,$$

for every $c > 0$.

Let $f(z)$ be any entire function and suppose that $\alpha(x) \in \Lambda$, $\beta(x) \in L^0$. Write

$$\begin{matrix}\rho(\alpha, \beta, f) \\ \lambda(\alpha, \beta, f)\end{matrix} = \lim_{r\to\infty} \begin{Bmatrix}\sup \\ \inf\end{Bmatrix} \frac{\alpha(\log M(r, f))}{\beta(\log r)}. \tag{1.3}$$

Then $\rho(\alpha, \beta, f)$ is called the generalized order of f and $\lambda(\alpha, \beta, f)$ the generalized lower order of f. If we take $\alpha(x) = \log x$, $\beta(x) = x$ we get the familiar definitions of order [3, p. 8; 14, pp. 32–34] and the lower order [16].

Let $\mu(r)$ denote the maximum term of an entire function $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $\nu(r)$ the rank of $\mu(r)$ and $M(r, f)$ the maximum modulus. In Theorem 1 we consider the expressions in (1.3) when $\log M(r, f)$ is replaced by $\log \mu(r)$ and by $\nu(r)$; and in Theorem 2 we obtain an inequality between the lower order $\lambda(\alpha, \beta, f)$ and an expression involving the coefficient a_n. In Theorem 3 we use these results to obtain expressions for $\rho(\alpha, \beta, f)$ and $\lambda(\alpha, \beta, f)$ involving the approximation error $E_n(f)$. We shall assume in Theorems 1–3 that f is not a constant function and that $\alpha(x) \in \Lambda$, $\beta(x) \in L^0$.

THEOREM 1. *Let $f(z)$ be entire. Set $F(x, c) = \beta^{-1}(c\alpha(x))$, $F(x, 1) = F(x)$. If for some function $\psi(x)$ tending to ∞ (howsoever slowly) as $x \to \infty$*

$$\beta(x\psi(x))/\beta(e^x) \to 0, \qquad \textit{as } x \to \infty, \tag{1.4}$$

and if

$$dF(x)/d(\log x) = O(1), \qquad \textit{as } x \to \infty, \tag{1.5}$$

then

$$\rho(\alpha, \beta, f) = \limsup_{r\to\infty} \frac{\alpha(\log \mu(r))}{\beta(\log r)} = \limsup_{r\to\infty} \frac{\alpha(\nu(r))}{\beta(\log r)}, \tag{1.6}$$

$$\lambda(\alpha, \beta, f) = \liminf_{r\to\infty} \frac{\alpha(\log \mu(r))}{\beta(\log r)} = \liminf_{r\to\infty} \frac{\alpha(\nu(r))}{\beta(\log r)}. \tag{1.7}$$

THEOREM 2. *Let*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \tag{1.8}$$

be an entire function. Then (i)

$$\liminf_{n\to\infty} \alpha(n)/\beta\left(\frac{1}{n} \log \frac{1}{|a_n|}\right) \leqslant \lambda(\alpha, \beta, f). \tag{1.9}$$

(ii) *Assume that* $|a_n/a_{n+1}|$ *is ultimately a nondecreasing function of* n *and* (1.4) *and* (1.5) *hold. Then there is an equality sign in* (1.9).

Remark. If

$$\frac{dF(x, c)}{d(\log x)} = O(1), \qquad x \to \infty, \tag{1.10}$$

for every $c > 0$, then Seremeta [11] has shown that

$$\rho(\alpha, \beta, f) = \limsup_{n\to\infty} \alpha(n)/\beta\left(\frac{1}{n}\log\frac{1}{|a_n|}\right). \tag{1.11}$$

THEOREM 3. *Let* $f(x) \in C[-1, 1]$ *and let* $E_n(f)$ *be defined by* (1.1). *Suppose* (1.2) *holds. Then* $f(x)$ *is the restriction to* $[-1, 1]$ *of an entire function* $f(z)$ *and* (i)

$$\lambda(\alpha, \beta, f) \geqslant \liminf_{n\to\infty} \alpha(n)/\beta\left(\frac{1}{n}\log\frac{1}{E_n(f)}\right). \tag{1.12}$$

(ii) *Assume also* (1.10) *and* (1.4). *Then*

$$\rho(\alpha, \beta, f) = \limsup_{n\to\infty} \alpha(n)/\beta\left(\frac{1}{n}\log\frac{1}{E_n(f)}\right). \tag{1.13}$$

(iii) *Assume further that* $E_n(f)/E_{n+1}(f)$ *is ultimately a nondecreasing function of* n. *Then there is an equality sign in* (1.12).

Remarks. Let $l_k x$ denote the kth iterate of the logarithm: $l_1 x = \log x$, $l_k x = \log(l_{k-1} x)$ $(k \geqslant 2)$.

(i) Let $\alpha(x) = \log x$, $\beta(x) = x$, $\psi(x) = l_k x$ $(k \geqslant 2)$. Then $F(x, 1) = \log x$, the hypotheses of Theorem 1 are satisfied and we get another proof of Whittaker's theorem [16, Theorem 1].

(ii) If we take $\alpha(x) = \log x$, $\beta(x) = x$, we see that Theorem 2 gives extensions of parts of Theorems 1 and 2 of [12].

(iii) Let $\alpha(x) = l_k x$, $\beta(x) = \log x$ $(k \geqslant 1)$ then $\alpha \in \Lambda$, $\beta \in L^0$ and Theorem 3 gives extensions of a theorem of Varga [15, Theorem 1] and some theorems of Reddy [7; Theorems 1, 2A, 2B]. Note that (1.13) holds whether $\rho(\alpha, \beta, f)$ is finite or infinite. If it is infinite then Theorem 3(ii) implies that the right-side expression in (1.13) is infinite and conversely. A similar remark applies to (1.6), (1.7), part (ii) of Theorem 2 and part (iii) of Theorem 3.

(iv) By appropriate choices of $\alpha(x)$, $\beta(x)$ we get some results, proved in [10, 13], from Theorems 1 and 2.

2. Proof of Theorem 1

Write

$$\begin{matrix}\theta_2 \\ \phi_2\end{matrix} = \lim_{r\to\infty} \begin{cases}\sup \\ \inf\end{cases} \frac{\alpha(\log \mu(r))}{\beta(\log r)},$$

$$\begin{matrix}\theta_3 \\ \phi_3\end{matrix} = \lim_{r\to\infty} \begin{cases}\sup \\ \inf\end{cases} \frac{\alpha(\nu(r))}{\beta(\log r)}.$$

We prove $\theta_2 = \theta_3$, $\phi_2 = \phi_3$ in parts (a), (b) and then complete the rest in (c) and (d). We shall abbreviate, in Theorems 1 and 2, $\rho(\alpha, \beta, f)$ to ρ and $\lambda(\alpha, \beta, f)$ to λ.

If f is a polynomial then $\theta_3 = \phi_3 = 0$, and using (1.4) we see that $\theta_2 = \phi_2 = \rho = \lambda = 0$. So we assume that f is a transcendental entire function.

(a) Since [3, pp. 12–13; 14, pp. 28–32]

$$\log \mu(2r) > \nu(r) \log 2,$$

and $\beta \in L^0$, we get $\theta_2 \geqslant \theta_3$, $\phi_2 \geqslant \phi_3$.

(b) To prove $\phi_2 \leqslant \phi_3$ we may assume that $\phi_3 < \infty$. Given $\epsilon > 0$, there exists an indefinitely increasing sequence $\{r_n\}$ such that for $n > n_0(\epsilon)$,

$$\alpha(\nu(r_n))/\beta(\log r_n) < \phi_3 + \epsilon. \tag{2.1}$$

Let $\psi_1(r) = \psi(\log \log r)$ and let $n_1 > n_0$ be so large that $\psi_1(r)$ is defined and positive for $r > r_{n_1}$. Let

$$E = \{r_n \mid n > n_1, F(\nu(r_n)) \leqslant (\log \log r_n)\, \psi_1(r_n)\}.$$

Then for $r = r_n \in E$,

$$\begin{aligned}\frac{\alpha(\log \mu(r)) - \alpha(\nu(r))}{\beta(\log r)} &< \frac{(1 + o(1))\, \alpha(\nu(r) \log r) - \alpha(\nu(r))}{\beta(\log r)} \\ &= \frac{(1 + o(1))\, \beta(F(\nu(r) \log r)) - \beta(F(\nu(r)))}{\beta(\log r)} \\ &< \frac{(1 + o(1))\, \beta(F(\nu(r)) + A \log \log r)}{\beta(\log r)} \\ &\leqslant \frac{(1 + o(1))\, \beta((1 + o(1))(\log \log r)\, \psi_1(r))}{\beta(\log r)} \\ &= o(1) \qquad \text{as} \qquad r = r_n (\in E) \to \infty.\end{aligned}$$

For $r = r_n \in CE$, $(n > n_1)$,

$$\frac{\alpha(\log \mu(r)) - \alpha(\nu(r))}{\beta(\log r)} \leqslant \frac{(1 + o(1))\beta\left(F(\nu(r))\left(1 + \frac{A \log\log r}{F(\nu(r))}\right)\right) - \alpha(\nu(r))}{\beta(\log r)}$$
$$= \frac{(1 + o(1))\,\beta(F(\nu(r)) - \alpha(\nu(r))}{\beta(\log r)}$$
$$= \frac{(1 + o(1))\,\alpha(\nu(r)) - \alpha(\nu(r))}{\beta(\log r)}$$
$$\leqslant o(1)(\phi_3 + \epsilon) = o(1)$$

and so $\phi_2 \leqslant \phi_3$. Note that if E (or cE) has only a finite number of elements then we need consider cE (resp. E) only. The above argument gives also $\theta_2 \leqslant \theta_3$, if we consider all $r > r_0(\epsilon)$ such that

$$\alpha(\nu(r))/\beta(\log r) < \theta_3 + \epsilon,$$

and define $E = \{r \mid r > r_1, F(\nu(r)) \leqslant (\log\log r)\, \psi_1(r)\}$. Hence $\theta_2 = \theta_3$, $\phi_2 = \phi_3$.

(c) Since $\alpha(\log M(r)) \geqslant \alpha(\log \mu(r))$, we have

$$\rho \geqslant \theta_2, \qquad \lambda \geqslant \phi_2.$$

(d) We now prove $\rho \leqslant \theta_2$, $\lambda \leqslant \phi_2$. Since [14, pp. 28–32]

$$M(r) \leqslant 3\mu(r)\nu\left(r + \frac{r}{\nu(r)}\right), \qquad r > r_0,$$
$$\log \mu(2er) > \nu(2r),$$

we have

$$\begin{aligned} &\log \nu(2r) = o(1)(\log \mu(2er)), \\ &\log M(r) \leqslant \log 3 + \log \mu(r) + \log \nu(2r) \leqslant (1 + o(1)) \log \mu(2er). \end{aligned} \tag{2.2}$$

Since $\beta \in L^0$, $\alpha \in \Lambda$ it follows that $\rho \leqslant \theta_2$. To prove $\lambda \leqslant \phi_2$, assume $\phi_2 < \infty$ and let in (2.2) $2er = r_n$ where $\{r_n\}$ is such that $\alpha(\log \mu(r_n))/\beta(\log r_n)$ tends to ϕ_2 as $n \to \infty$. Then (2.2) implies that $\lambda \leqslant \phi_2$. The theorem is proved.

3. Proof of Theorem 2

Write

$$\begin{matrix}\rho_0 \\ \lambda_0\end{matrix} = \lim_{n\to\infty} \begin{Bmatrix}\sup \\ \inf\end{Bmatrix} \alpha(n)/\beta\left(\frac{1}{n} \log \frac{1}{|a_n|}\right).$$

(i) If f is a polynomial then $\rho_0 = \lambda_0 = \rho = \lambda = 0$ and so we assume f to be a transcendental entire function. We prove $\lambda_0 \leqslant \lambda$. We may assume $\lambda < \infty$. Given $\epsilon > 0$, there exists a sequence $\{r_n\}_1^\infty$ such that for $r = r_n$,

$$\alpha(\log M(r)) < (\lambda + \epsilon)\,\beta(\log r),$$

that is

$$M(r) < \exp\{\alpha^{-1}((\lambda + \epsilon)\,\beta(\log r))\}.$$

By Cauchy inequality

$$|\,a_k\,| \leqslant M(r)/r^k,$$

we get for $r = r_n$ and each $k \geqslant 0$,

$$|\,a_k\,| < \exp\{\alpha^{-1}((\lambda + \epsilon)\,\beta((\log r))\}/r^k.$$

Choose $k = [\alpha^{-1}((\lambda + \epsilon)\,\beta((\log r_n))]$, where $[x]$ denotes the integer part of x. Then

$$\alpha^{-1}((\lambda + \epsilon)\,\beta(\log r_n)) - 1 < k \leqslant \alpha^{-1}((\lambda + \epsilon)\,\beta(\log r_n)), \qquad (3.1)$$

and

$$|\,a_k\,| \leqslant e^{(k+1)}/e^{k \log r_n}.$$

Hence

$$\frac{1}{k}\log\frac{1}{|\,a_k\,|} \geqslant (\log r_n - 1)(1 + o(1)) = (1 + o(1))\log r_n\,,$$

$$\alpha(k)/\beta\left(\frac{1}{k}\log\frac{1}{|\,a_k\,|}\right) \leqslant (1 + o(1))\,\alpha(k)/\beta(\log r_n) \leqslant (1 + o(1))(\lambda + \epsilon),$$

where we have used (3.1). Hence $\lambda_0 \leqslant \lambda$.

(ii) Set $\xi(n) = |\,a_n/a_{n+1}\,|$. Then $\xi(n) \to \infty$ and $\xi(n) > \xi(n-1)$ for an infinity of n (see cf. [13]). When $\xi(n) > \xi(n-1)$, we have $\mu(r) = |\,a_n\,|\,r^n$, $\nu(r) = n$ for $\xi(n-1) \leqslant r < \xi(n)$.

Given $\epsilon > 0$, write $\boldsymbol{\lambda} = \lambda - \epsilon$ if $\lambda < \infty$, $\boldsymbol{\lambda} = H$ (an arbitrarily large constant) if $\lambda = \infty$. Then for $r > R_0 = R_0(\epsilon)$, $\nu(r) > \alpha^{-1}(\boldsymbol{\lambda}\beta(\log r))$. Let $|\,z\,| = r > R_0$ and let $a_{m_1}z^{m_1}$, $a_{m_2}z^{m_2}$, ($\xi(m_1 - 1) > R_0$) be two consecutive maximum terms. Then $m_1 \leqslant m_2 - 1$. Let $m_1 < n \leqslant m_2$. Then $\nu(r) = m_1$ for $\xi(m_1 - 1) \leqslant r < \xi(m_1)$. So

$$m_1 = \nu(r) > \alpha^{-1}(\boldsymbol{\lambda}\beta(\log r)) \geqslant \alpha^{-1}(\boldsymbol{\lambda}\beta(\log(\xi(m_1) - d)))$$

where d is a constant such that $0 < d < \min\{1, (\xi(m_1) - \xi(m_1 - 1))/2\}$. Further $\xi(m_1) = \xi(m_1 + 1) = \cdots = \xi(n-1)$. Hence (writing $a(m)$ for a_m)

$$\xi(n_0 + 1)\cdots\xi(n-1) = \left|\frac{a(n_0 + 1)}{a(n)}\right| \leqslant (\xi(n-1))^{n-n_0-1}$$

and so

$$\frac{1}{n}\log\frac{1}{|a_n|} < o(1) + \log \xi(m_1) \leqslant (1 + o(1))\, \beta^{-1}(\alpha(m_1)/\lambda)$$
$$\leqslant (1 + o(1))\, \beta^{-1}(\alpha(n)/\lambda).$$

Consequently

$$\lambda \leqslant (1 + o(1))\, \alpha(n)/\beta\left(\frac{1}{n}\log\frac{1}{|a_n|}\right)$$

and so $\lambda \leqslant \lambda_0$. The proof is complete.

4. Proof of Theorem 3

Denote the expression on the right of (1.13) by ρ_0 and that on the right of (1.12) by λ_0. By hypothesis (1.2) $f(z)$ and $g(z) = \sum_{n=0}^{\infty} E_n(f)\, z^n$ are entire functions. As in Theorem 2, we may assume that f is not a polynomial. This assumption implies that g is not a polynomial. Now [5, p. 76–78; 15] for $r > 1$ and $n \geqslant 0$,

$$E_n(f) \leqslant \frac{2B(r)}{r^n(r-1)},$$

where

$$M\left(\frac{r^2-1}{2r}, f\right) \leqslant B(r) \leqslant M\left(\frac{r^2+1}{2r}, f\right), \qquad r > 1. \tag{4.1}$$

Consequently for $r \geqslant 3$ and $n \geqslant 0$

$$E_n(f) \leqslant M(r, f)/r^n. \tag{4.2}$$

Further

$$B(r) \leqslant c_0 + 2r \sum_{k=0}^{\infty} E_k(f)\, r^k,$$

where c_0 is a positive constant. Hence for $r \geqslant 3$,

$$M\left(\frac{r}{3}, f\right) \leqslant c_0 + 2rM(r, g)$$

and so for all large r

$$M(r, f) \leqslant 9rM(3r, g)$$

and

$$\frac{\alpha(\log M(r, f))}{\beta(\log r)} \leqslant \frac{\alpha((1 + o(1)) \log M(3r, g))}{\beta(\log r)}.$$

Since $\beta \in L^0$, we have

$$\lambda(\alpha, \beta, f) \leqslant \lambda(\alpha, \beta, g); \qquad \rho(\alpha, \beta, f) \leqslant \rho(\alpha, \beta, g). \tag{4.3}$$

(i) We prove (1.12). Assume, as we may, that $\lambda_0 > 0$. Write $\boldsymbol{\lambda} = \lambda_0 - \epsilon$ if $\lambda_0 < \infty$, $\boldsymbol{\lambda} = H$ if $\lambda_0 = \infty$. Then for all $n > n_0$,

$$E_n(f) < 1, \qquad \alpha(n)/\boldsymbol{\lambda} > \beta\left(\frac{1}{n}\log\frac{1}{E_n(f)}\right),$$

that is,

$$E_n(f) > 1\Big/\exp\left\{n\beta^{-1}\left(\frac{\alpha(n)}{\boldsymbol{\lambda}}\right)\right\}.$$

Let $r_n = \exp\{1 + \beta^{-1}(\alpha(n)/\boldsymbol{\lambda})\}$. By (4.2) we have for $r_n \leqslant r \leqslant r_{n+1}$ ($n > n_0$, $r > 3$)

$$M(r, f) \geqslant r^n E_n(f) \geqslant r_n{}^n E_n(f) > \exp n,$$

and

$$\frac{\alpha(\log M(r, f))}{\beta(\log r)} \geqslant \frac{\alpha(n)}{\beta(\log r_{n+1})} = \frac{(1 + o(1))\, \alpha(n)\boldsymbol{\lambda}}{\alpha(n + 1)}.$$

Hence

$$\lambda_0 \leqslant \lambda(\alpha, \beta, f). \tag{4.4}$$

(ii.a) By (4.2) we have for $r \geqslant 3$

$$\mu(r, g) \leqslant M(r, f); \tag{4.5}$$

and (1.6), (1.7), (4.3), and (4.5) imply that

$$\lambda(\alpha, \beta, f) = \lambda(\alpha, \beta, g); \qquad \rho(\alpha, \beta, f) = \rho(\alpha, \beta, g). \tag{4.6}$$

(ii.b) Since [11; Theorem 1] $\rho(\alpha, \beta, g) = \rho_0$, (1.13) follows from (4.6).

(iii) By Theorem 2(ii) and (4.6), $\lambda_0 = \lambda(\alpha, \beta, g) = \lambda(\alpha, \beta, f)$.

The proof is complete.

5. Theorems 4 and 5

In what follows we extend $\beta(x)$ over $(-\infty, a)$ so that $\beta(x)$ is nonnegative, nondecreasing, and continuous over $(-\infty, a]$. (The constant a in (H, i) is throughout a positive number.) We assume (1.4) and (1.10). (The condition (1.4) assures that the growth of β is not "too slow.") We denote by $\{n_k\}$ a strictly increasing sequence of positive integers. For convenience of notation we sometimes write $a(n)$ for a_n.

THEOREM 4. *Let f be a transcendental entire function defined by* (1.8) *and let $E = E(f)$ denote the sequence of positive integers $\{n_k\}_1^\infty$ such that* max $\{|a(n_{k-1})|, |a(n_k)|\} > 0$ *for $k = 2, 3, \ldots$. Then*

$$\lambda(\alpha, \beta, f) = \sup_{\{n_k\}} \left\{ \liminf_{k\to\infty} \alpha(n_{k-1}) \Big/ \beta \left(\frac{1}{n_k} \log \frac{1}{|a(n_k)|} \right) \right\}, \tag{5.1}$$

$$\lambda(\alpha, \beta, f) = \sup_{\{n_k\}\in E} \left\{ \liminf_{\substack{k\to\infty \\ \{n_k\}\in E}} \alpha(n_{k-1}) \Big/ \beta \left(\frac{1}{n_k - n_{k-1}} \log \left| \frac{a(n_{k-1})}{a(n_k)} \right| \right) \right\}, \tag{5.2}$$

where supremum, in (5.1), *is taken over all sequences $\{n_k\}$, and in* (5.2) *over all sequences $\{n_k\} \in E$.*

Proof. Denote by $\lambda_0 = \lambda_0(\{n_k\})$ the expression in curly brackets on the right in (5.1) and by $\lambda_1 = \lambda_1(\{n_k\})$ the similar expression in (5.2). Write $\lambda = \lambda(\alpha, \beta, f)$.

(i) *If $\{n_k\} \in E$ then $\lambda_1(\{n_k\}) \leqslant \lambda_0(\{n_k\})$.*

To prove (i) we may suppose $\lambda_1 > 0$. Then $|a(n_k)| > 0$ for $n_k \in E, k > k_0$. Write $\xi = \lambda_1 - \epsilon$ if $\lambda_1 < \infty$, $\xi = H$ if $\lambda_1 = \infty$. Then for $N_0 < N < M$,

$$\sum_{k=N}^{M} \log \left| \frac{a(n_{k-1})}{a(n_k)} \right| < \sum_{k=N}^{M} (n_k - n_{k-1})\, \beta^{-1} \left(\frac{\alpha(n_{k-1})}{\xi} \right)$$
$$< \beta^{-1} \left(\frac{\alpha(n_{M-1})}{\xi} \right) (n_M - n_{N-1}).$$

Hence

$$(1 + o(1)) \left(\frac{1}{n_M} \log \frac{1}{|a(n_M)|} \right) < \beta^{-1} \left(\frac{\alpha(n_{M-1})}{\xi} \right),$$

and (i) follows.

(ii) *If $\{n_k\}$ is the range of $\nu(r, f)$, then $\lambda = \lambda_1(\{n_k\})$.*

Let

$$\eta(n_k) = \left| \frac{a(n_{k-1})}{a(n_k)} \right|^{1/(n_k - n_{k-1})} \qquad (k > k_0).$$

Then for $\eta(n_k) \leqslant r < \eta(n_{k+1})$, $\mu(r, f) = |a(n_k)|\, r^{n_k}$, $\nu(r, f) = n_k$. Further $\eta(n_k) < \eta(n_k + 1) = \cdots = \eta(n_{k+1})$. In the interval $(\eta(n_k), \eta(n_k + 1))$, $\alpha(\nu(r))/\beta(\log r) \downarrow$ and

$$\lambda = \liminf_{r\to\infty} \frac{\alpha(\nu(r))}{\beta(\log r)} = \liminf_{k\to\infty} \frac{\alpha(n_k)}{\beta\{\log(\eta(n_k + 1) - h)\}}$$

where $h > 0$ is sufficiently small. We now use (H, ii) and obtain (ii).

(iii) *For any sequence $\{n_k\}$, $\lambda_0(\{n_k\}) \leqslant \lambda$.*

The proof is similar to that of Theorem 2(i) and omitted.
The theorem follows from (i)–(iii).

THEOREM 5. *Let* $f(x) \in C[-1, 1]$ *and suppose that* $f(x)$ *is not a polynomial and* (1.2) *holds. Then* $f(x)$ *is the restriction to* $[-1, 1]$ *of an entire function* $f(z)$ *and*

$$\lambda(\alpha, \beta, f) = \sup_{\{n_k\}} \left\{ \liminf_{k\to\infty} \alpha(n_{k-1}) \Big/ \beta \left(\frac{1}{n_k} \log \frac{1}{E(n_k)} \right) \right\}$$
$$= \sup_{\{n_k\}} \left\{ \liminf_{k\to\infty} \alpha(n_{k-1}) \Big/ \beta \left(\frac{1}{n_k - n_{k-1}} \log \left| \frac{E(n_{k-1})}{E(n_k)} \right| \right) \right\}.$$

The proof follows immediately from (4.6) and Theorem 4 and is omitted.

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